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On the M-projective curvature tensor of Sasakian manifolds

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Abstract

This paper is an analysis of the properties of the $M-$projective curvature tensor in Sasakian, Einstein Sasakian and $\eta-$Einstein Sasakian manifolds.

Key words: Sasakian manifolds; M-projective curvature tensor; η-Einstein manifold.

Introduction

In 1971, Pokhariyal and Mishra[1] defined a tensor field W^* on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \quad (1.1)$$

where
$W^*(X,Y,Z,U) = W^*(Z,U,X,Y)$,
such a tensor field W^* is known as M-projective curvature tensor.

The properties of the M-projective curvature tensor in Sasakian and Kahler manifolds were studied by Ojha.[2, 3] He showed that it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor. The author[4] proved that an *M*-

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The properties of the M-projective curvature tensor in Sasakian and Kahler manifolds were studied by Ojha.[2, 3] He showed that it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor. The author[4] proved that an *M*-projectively flat Para-Sasakian manifold is an Einstein manifold. He has alsoshown that if in an Einstein P-Sasakian manifold $R(\xi,X)W^* = 0$ holds, then itis locally isometric with a unit sphere $H^n(1)$. Also, an n-dimensional η-EinsteinP-Sasakian manifold satisfies $W^*(\xi,X)R = 0$ if and only if either the manifold islocally isometric to the hyperbolic space $H^n(-1)$ or the scalar curvature tensor *r* of the manifold is *n*(*n*-1). Recently M-projective curvature tensor is studied by many Geometers such as Chaubey and Ojha,[5] Singh,[6] Bagewadi *et al.*[7] etc.

This paper deals with some properties of M-projective curvature tensor in Sasakian manifolds M_n.

PRELIMINARIES

Let M_n be an $n = (2m+1)$ -dimensional almost contact metric manifold equipped with an almost contact metric structure (φ, η, ξ, g) consisting of a (1, 1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y.

Form (2.1) and (2.2), it can be easily seen that

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X). \tag{2.3}$$

An almost contact metric manifold M_n is said to be

(a) a contact metric manifold if

$$g(X, \varphi Y) = d\eta(X, Y), \tag{2.4}$$

(b) a K-contact manifold if the vector field ξ is killing equivalently

$$D_X\xi = -\varphi X, \tag{2.5}$$

where D is Riemannian connection and

(c) a Sasakian manifold if

$$(D_X\varphi)Y = g(X, Y)\xi - \eta(Y)X. \tag{2.6}$$

A K-contact manifold is a contact metric manifold while the converse is true if the Lie derivative of φ in the characteristic direction ξ vanishes. A 3-dimensional manifold is a Sasakian manifold.

It is well known that a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

In a Sasakian manifold equipped with the structures(φ, η, ξ, g), the following relations also hold[8, 9]

$$(D_X\eta)Y = g(X, \varphi Y), \tag{2.8}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{2.10}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.11}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.13}$$

for all vector fields X, Y, Z where R is Riemannian curvature tensor and S is Ricci tensor.

A Sasakian manifold M_n is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = a\, g(X, Y) + b\, \eta(X)\eta(Y) \tag{2.14}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on (M_n, g) . If $b = 0$ then η- Einstein manifold becomes Einstein manifold.

In view of (2.1) and (2.14), we have

$$QX = a\, X + b\, \eta(X)\xi, \tag{2.15}$$

where$Q$ is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

Again, contracting (2.15) with respect to X and using (2.1), we have

$$r = na + b \tag{2.16}$$

Now, substituting $X = \xi$and $Y = \xi$in (2.14) and then using (2.1) and (2.11), we obtain

$$a + b = n - 1. \tag{2.17}$$

Equations (2.16) and (2.17) give

$$a = \frac{r}{n-1} - 1 \quad \text{and} \quad b = -\left(\frac{r}{n-1} - n\right). \tag{2.18}$$

Sasakian manifolds satisfying $W^* = 0$

In view of $W^* = 0$, (1.1) becomes

$$R(X, Y)Z = \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \tag{3.1}$$

Replacing Z by ξ in (3.1) and then using (2.1) and (2.11),we obtain

$$(n-1)(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY.$$

Again putting $Y = \xi$ in the above relation and using (2.1) and (2.11), we obtain

$$QX = (n-1)X \Leftrightarrow \quad S(X, Y) = (n-1)g(X, Y) \tag{3.2}$$

and

$$r = n(n-1).$$

In consequence of (3.2), (3.1) becomes

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{3.3}$$

which shows that m-projectively flat Sasakian manifold is of constant curvature. The value of this constant is +1. Hence we can state

Theorem3.1:An n- dimensional Sasakian manifold M_n is m-projectively flat if and only if it has constant curvature +1.

Theorem 3.2: An n-dimensional Sasakian manifoldM_n is m-projectively flat if and only if it is locally isometric to a unit sphere $\mathbb{S}^n(1)$.

An Einstein Sasakian manifold satisfying $R(\xi, X).W^* = 0$

Theorem 4.1: An Einstein Sasakian manifold M_n satisfies $R(\xi, X).W^* = 0$ if and only if it is locally isometric a unit sphere $S^n(1)$.

Proof: Let the Sasakian manifold be Einstein i.e. $S(X,Y) = kg(X,Y)$, then

$$W^*(X,Y)Z = R(X,Y)Z - \frac{k}{(n-1)}[g(Y,Z)X - g(X,Z)Y] \quad (4.1)$$

In view of (2.1), (2.10) and (4.1), we have

$$\eta(W^*(X,Y)Z) = \left(1 - \frac{k}{n-1}\right)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \quad (4.2)$$

Replacing Z by ξ and using (2.1) in above equation, we have

$$\eta(W^*(X,Y)\xi) = 0. \quad (4.3)$$

Now,

$$R(X,Y).W^*(Z,U)V = R(X,Y)W^*(Z,U)V - W^*(R(X,Y)Z,U)V - W^*(Z,R(X,Y)U)V - W^*(Z,U)R(X,Y)V$$

Or

$$R(X,Y)W^*(Z,U)V - W^*(R(X,Y)Z,U)V - W^*(Z,R(X,Y)U)V - W^*(Z,U)R(X,Y)V = 0.$$

Taking inner product of the above equation with ξ, we obtain

$$g(R(X,Y)W^*(Z,U)V,\xi) - g(W^*(R(X,Y)Z,U)V,\xi) - g(W^*(Z,R(X,Y)U)V,\xi) - g(W^*(Z,U)R(X,Y)V,\xi) = 0. \quad (4.4)$$

Replacing $X = \xi$ in the equation (4.4), we get

$$g(R(\xi,Y)W^*(Z,U)V,\xi) - g(W^*(R(\xi,Y)Z,U)V,\xi) - g(W^*(Z,R(\xi,Y)U)V,\xi) - g(W^*(Z,U)R(\xi,Y)V,\xi) = 0. \quad (4.5)$$

Using (2.9), (2.10) and (2.11) in the above equation, we get

$$W^*(Z,U,V,Y) - \left(1 - \frac{k}{n-1}\right)[\eta(Y)\{\eta(Z)g(U,V) - \eta(U)g(Z,V)\} - \eta(Z)\{\eta(Y)g(U,V) - \eta(U)g(Y,V)\} - \eta(U)\{\eta(Z)g(Y,V) - \eta(Y)g(Z,V)\} - \eta(V)\{\eta(Z)g(U,Y) - \eta(U)g(Z,Y)\} + g(Y,Z)\{g(U,V) - \eta(U)\eta(V)\} + g(Y,U)\{\eta(V)\eta(Z) - g(V,Z)\}] = 0.$$

Or,

$$W^*(Z,U,V,Y) = \left(1 - \frac{k}{n-1}\right)[g(Y,Z)g(U,V) - g(Y,U)g(V,Z)].$$

which implies that

$$W^*(Z,U,V) = \left(1 - \frac{k}{n-1}\right)\{g(U,V)Z - g(Z,V)U\}. \quad (4.6)$$

In view of (4.1) and (4.6), we have

$$R(Z,U,V) = \{g(U,V)Z - g(Z,V)U\}. \quad (4.7)$$

This completes the proof.

Contracting (4.7) with respect to Z, we obtain

$$S(U,V) = (n-1)g(U,V) \quad (4.8)$$

and

$$QU = (n-1)U \quad (4.9)$$

which gives

$r = n(n-1)$.

In consequences of (1.1), (4.7), (4.8) and (4.9), we have

$W^*(X,Y)Z = 0$.

Hence, we can say

Theorem 4.2: An Einstein Sasakian manifold M_n satisfies $R(\xi, X).W^* = 0$ if and only if it is m-projectively flat.

In view of the theorems (4.1) and (4.2), we can state that

Corollary 4.3: An Einstein Sasakian manifold M_n satisfies $R(\xi, X).W^* = 0$ if and only if it is either M_n is m-projectively flat or it is locally isometric to a unit sphere $\mathbb{S}^n(1)$.

η - Einstein Sasakian manifolds satisfying $W^*(\xi, X)R = 0$

Replacing X by ξ in (1.1) and then using (2.1), (2.9), (2.14), (2.15) and (2.18), we obtain

$$W^*(\xi,Y)Z = k\{g(Y,Z)\xi - \eta(Z)Y\}, \quad (5.1)$$

Where $k = \left\{1 - \frac{1}{2(n-1)}\left(\frac{r}{n-1} + n - 2\right)\right\}$.

We know that

$$(W^*(\xi,X).R)(Y,Z)U = (W^*(\xi,X)R(Y,Z)U$$

$$-R(W^*(\xi,X)Y,Z)U - R(Y,(W^*(\xi,X)Z)U - R(Y,Z)(W^*(\xi,X)U.$$

Using $(W^*(\xi,X).R = 0$ in the above relation, we get

$$(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U - R(Y,(W^*(\xi,X)Z)U - R(Y,Z)(W^*(\xi,X)U = 0.$$

In view of (5.1), last result becomes

$$k[R(Y,Z,U,X)\xi - \eta(Y)g(Z,U)X + \eta(Z)g(Y,U)X - R(g(X,Y)\xi - \eta(Y)X,Z)U + R(Y,g(X,Z)\xi - \eta(Z)X)U - R(Y,Z)(g(X,U)\xi - \eta(U)X)] = 0. \quad (5.2)$$

Taking inner product of the equation (5.2) with ξ and using the equations (2.9) and (2.10), we obtain

$$R(Y,Z)U = g(Z,U)Y - g(Y,U)Z. \quad (5.3)$$

Contracting (5.3) with respect to Y, we get

$$S(Z,U) = (n-1)g(Z,U)$$

Or

$$QZ = (n-1)Z. \quad (5.4)$$

Again contracting (5.4), we have

$$r = n(n-1).$$

Conversely, if M_n is locally isometric to a unit sphere $\mathbb{S}^n(1)$ or M_n has a scalar curvature $r = n(n-1)$ then from (5.1), it follows that M_n is m-projectively flat.

Thus, we can state

Theorem: 5.1: An n-dimensional η -Einstein Sasakian manifold M_n satisfies $W^*(\xi,X)R = 0$ if and only if either M_n is is locally isometric to a unit sphere $\mathbb{S}^n(1)$ or M_n is m-projectively flat.

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