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On pseudo $\widetilde{W}_2$ flat LP-Sasakian Manifold with a coefficient α

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ABSTRACT

De, Shaikh and Sengupta introduced the notion of LP-Sasakian manifolds with coefficient α which generalized the notion of LP-Sasakian manifolds. Recently, Ikawa and his coauthors studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. The object of the paper is to steady pseudo $\widetilde{W}_2$ flat LP-Sasakian manifolds with coefficient α.

Key words: LP-Sasakian manifold; Lorentzian metric; Coefficient α.

INTRODUCTION

Let M be the n-dimensional differential manifold endowed with a (1,1) tensor field ϕ,a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type (0,2) such that for each $p \epsilon M$, the tensor $g_p: T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,+,\dots,+)$, where T_pM denote the tangent vector space of M at p and R is the real number space, which satisfies

$$\eta(\xi) = -1\,,\ \phi^2 X = X + \eta(X)\xi, \qquad \dots (1.1)$$

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \qquad \dots (1.2)$$

for all vector field *X, Y.* Then such a structure (ϕ,ξ,η,g) is called Lorentzian almost paracontact manifold.[2] In the Lorentzian almost paracontact manifold *M*, the following relations holds[2]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \qquad \dots (1.3)$$

$$\Omega(X,Y) = \Omega(Y,X), \text{ where } \Omega(X,Y) = g(X,\phi Y). \qquad \dots (1.4)$$

In the Lorentzian almost paracontact manifold *M,* if the relation

$$\nabla_Z \Omega(X,Y) = \alpha[\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) + \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)],\ (\alpha \neq 0) \qquad \dots (1.5)$$

$$\text{and} \quad \Omega(X,Y) = \frac{1}{\alpha}(\nabla_X \eta)Y, \qquad \dots (1.6)$$

holds, where ∇ denote the operator of covariant differentiation with respect to the Lorentzian metric g, then M is called an LP-Sasakian manifold with a coefficient α.[1] An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold.[2]

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If a vector field V satisfies the equation of the following form:

$$\nabla_X V = \beta X + T(X)V$$

where β a non-zero scalar function and T is is a covariant vector field, then V is called a torse-forming vector field.[6]

In a Lorentzian manifold M, if we assume that ξ is a unit torse-forming vector field, then we have the following:

$$(\nabla_X \eta)Y = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \quad \ldots(1.7)$$

where α is non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (1.7) is an LP-Sasakian manifold with a coefficient α. And, if η satisfy

$$(\nabla_X \eta)Y = \varepsilon\{g(X,Y) + \eta(X)\eta(Y)\}, \varepsilon^2 = 1 \ldots (1.8)$$

then M is called an LSP-Sasakian manifold.[2] In particular, if α satisfies (1.7) and the equation of the following form:

$$\alpha(X) = \rho\eta(X), \qquad \alpha(X) = \nabla_X \alpha, \quad \ldots(1.9)$$

where ρ is a scalar function, then ξ is called a concircular vector field.

If we put

$$\phi X = \frac{1}{\alpha}(\nabla_X \xi), \quad \ldots(1.10)$$

we can easily find that

$$\phi^2 X = X + \eta(X)\xi.$$

Hence M is a manifold with a Lorentzian almost paracontact structure (ϕ, ξ, η, g). Such a manifold M is called a Lorentzian almost paracontact manifold with a structure of the concircular type.[1]

Let us consider an LP-Sasakian manifold M (ϕ, ξ, η, g) with a coefficient α.

Then we have the following relation.[1]

$$\eta(R(X,Y)Z) = -\alpha(X)\Omega(Y,Z) + \alpha(Y)\Omega(X,Z) + \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \quad \ldots(1.11)$$

and $$S(X,\xi) = -\psi\alpha(X) + (n-1)\alpha^2\eta(X) + \alpha(\phi X), \quad \ldots(1.12)$$

where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold and $\psi = trace(\phi)$.

Lemma 1.1. In an LP-Sasakian manifold with a non coefficient α, one of the following cases occur,[1]:

(i) $\psi^2 = (n-1)^2$, (ii) $\alpha(Y) = -\rho\eta(Y)$, where $\rho = \alpha(\xi)$.

Lemma 1.2. In a Lorentzian almost paracontact manifold $M(\phi, \xi, \eta, g)$ with its structure (ϕ, ξ, η, g) satisfying $\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y)$, where α is a non zero scalar function, the vector field ξ is torse-forming if and only if the relation $\psi^2 = (n-1)^2$ holds good.[1]

Pseudo $\widetilde{W}_2$ curvature tensor on a Riemannian manifold $(M,g)(n > 1)$ of type (1, 3) is defined as follows [7]

$$\widetilde{W}_2(X,Y)Z = a\,R(X,Y)Z + b[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{(n-1)} + b\right)[g(Y,Z)X - g(X,Z)Y]$$

where a and b are constant such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci tensor, r is the scalar curvature and Q is the *(1,1)* Ricci tensor defined by

$S(X,Y) = g(QX,Y)$, for all X and Y.

2. Pseudo $\widetilde{W}_2$ flat LP-Sasakian manifold with a coefficient α

Let us consider a pseudo $\widetilde{W}_2$flat LP-Sasakian manifold M with a coefficient α. First suppose that α is non constant. Then since the pseudo $\widetilde{W}_2$ curvature tensor vanished, the curvature tensor $'R$ satisfies

$'R(X,Y,Z,W) = -\frac{b}{a}[S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] + \frac{r}{n}\left(\frac{1}{(n-1)} + \frac{b}{a}\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$

which gives on using the identity $'R(X,Y,Z,W) = -'R(X,Y,W,Z)$

$$-'R(X,Y,W,Z) = -\frac{b}{a}[S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] + \frac{r}{n}\left(\frac{1}{(n-1)} + \frac{b}{a}\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \quad \text{... (2.1)}$$

Putting $Z = \xi$ in (2.1) and then using (1.11) and (1.12), we get

$$-\alpha(X)\Omega(Y,W) + \alpha(Y)\Omega(X,W) + \alpha^2\{g(Y,W)\eta(X) - g(X,W)\eta(Y)\} = \frac{b}{a}[S(X,W)\eta(Y) - S(Y,W)\eta(X)] - \frac{r}{n}\left(\frac{1}{(n-1)} + \frac{2b}{a}\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \quad \text{... (2.2)}$$

Again on putting $X = \xi$ in (2.2) and using (1.3) and (1.12), we obtain

$$S(Y,W) = \left[-\frac{a}{b}\alpha^2 + r\left(\frac{a+b(n-1)}{n(n-1)b}\right)\right]g(Y,W) + \left[-\alpha^2\left(\frac{a+b(n-1)}{b}\right) + r\left(\frac{a+b(n-1)}{n(n-1)b}\right)\right]\eta(Y)\eta(W) + \{\psi\alpha(W) - \alpha(\phi W)\}\eta(Y) - \frac{a}{b}\rho\ \Omega(Y,W). \quad \text{...(2.3)}$$

where $\rho = \alpha(\xi)$.

We now suppose that M is η −Einstein. If an LP-Sasakian manifold M with the coefficient α satisfies the relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are the associated functions on the manifold. Then the manifold M is called an η −Einstein. Then we have [1]

$$S(Y,W) = \left[\frac{r}{n-1} - \alpha^2 - \frac{\psi\rho}{n-1}\right]g(Y,W) + \left[\frac{r}{n-1} - n\alpha^2 - \frac{n\psi\rho}{n-1}\right]\eta(Y)\eta(W) \quad \text{...(2.4)}$$

Putting $X = Y = e_i$ in (2.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over $1 \le i \le n$, we get

$$r = n(n-1)\alpha^2 + n\rho\psi\ . \quad \text{...(2.5)}$$

By virtue of (2.4) and (2.3), we get

$$\left[\left(\frac{a-b}{b}\right)\alpha^2 + r\left(\frac{b-a}{n(n-1)b}\right) - \frac{\psi\rho}{(n-1)}\right]g(Y,W) + \left[\left(\frac{a-b}{b}\right)\alpha^2 + r\left(\frac{b-a}{n(n-1)b}\right) - \frac{n\psi\rho}{(n-1)}\right]\eta(Y)\eta(W) - \{\psi\alpha(W) - \alpha(\phi W)\}\eta(Y) + \frac{a}{b}\rho\ \Omega(Y,W) = 0 \quad \text{...(2.6)}$$

Putting $Y = \xi$ in (2.6) we get

$$\psi\alpha(W) - \alpha(\phi W) = -\psi\,\rho\,\eta(W)$$

For all W. Replacing W by Y in above equation, we get

$$\psi\alpha(Y) - \alpha(\phi Y) = -\psi\,\rho\,\eta(Y) \quad \text{...(2.7)}$$

Using (2.7) in (2.6) and then by virtue of (2.5), we get

$$\rho\frac{a}{b}\left[-\frac{\psi}{n-1}\{g(Y,W) + \eta(Y)\eta(W)\} + \Omega(Y,W)\right] = 0 \quad \text{...(2.8)}$$

If $\rho = 0$, then from (2.7) we have $\alpha(\phi Y) = \psi\alpha(Y)$. Thus since ψ is an eigenvalue of the matrix(ϕ), ψ is equal to± 1. Hence, by virtue of Lemma1.1, we get $\alpha(Y) = 0$ for all Y and so α is constant, which contradicts to our assumption.

Consequently, we have $\rho \neq 0$ and hence from (2.8) we get

$$\frac{a}{b}\left[-\frac{\psi}{n-1}\{g(Y,W) + \eta(Y)\eta(W)\} + \Omega(Y,W)\right] = 0. \quad \text{...(2.9)}$$

Putting $Y = \phi Y$ in (2.9) and then using (1.3), we obtain

$$\frac{a}{b}\Big[-\frac{\psi}{n-1}\Omega(Y,W)+ \{g(Y,W)+\eta(Y)\eta(W)\}\Big]=0.$$
...(2.10)

Combining (2.9) and (2.10), we get

$$[\psi^2-(n-1)^2]\{g(Y,W)+\eta(Y)\eta(W)\}=0,$$

which gives by virtue of $n>1$,

$$\psi^2=(n-1)^2. \qquad ...(2.11)$$

Hence Lemma 1.2 proves that ξ is torse-forming.

We have $(\nabla_X\eta)(Y)=\beta\{g(X,Y)+\eta(X)\eta(Y)\}$.

Then from (1.6) we get

$$\Omega(X,Y)=\frac{\beta}{\alpha}\{g(X,Y)+\eta(X)\eta(Y)\} = g\left\{\frac{\beta}{\alpha}(X+\eta(X)\xi),Y\right\}$$

and $\Omega(X,Y)=g(\phi X,Y)$.

Since is g non-singular, we have

$$\phi(X)=\left(\frac{\beta}{\alpha}\right)(X+\eta(X)\xi)$$

and $\phi^2(X)=\left(\frac{\beta}{\alpha}\right)^2(X+\eta(X)\xi)$.

It follows from (1.1) that $\left(\frac{\beta}{\alpha}\right)^2=1$ and hence $\alpha=\pm\beta$. Thus we have

$$\phi(X)=\pm(X+\eta(X)\xi).$$

By virtue of (2.7) we have, $\alpha(Y)=-\rho\eta(Y)$, where $\rho=(\xi)$. Thus, we conclude that ξ is a concircular vector field. Than we have the equation of the following form:

$$(\nabla_X\eta)(Y)=\beta\{g(X,Y)+\eta(X)\eta(Y)\},$$

where β is a certain function and $\nabla_X\beta=\sigma\eta(X)$ for a certain scalar function σ.

Hence by virtue of (1.6) we have $\alpha=\pm\beta$. Thus

$$\Omega(X,Y)=\epsilon\{g(X,Y)+\eta(X)\eta(Y)\},\ \epsilon^2=1,$$
$$\psi=\epsilon(n-1),\ \nabla_X\alpha=\alpha(X)=\rho\eta(X),\ \rho=\epsilon\sigma.$$

Using these relations in (2.3) and (2.7), it can be easily seen that M is η −Einstein.

Thus we can state the following:

Theorem 2.1 In a pseudo $\widetilde{W}_2$flat LP-Sasakian manifold $M\,(n>1)$ with a non-constant coefficient α, the characteristic vector field ξ is a concircular vector field if and only if M is η −Einstein.

Next we consider the case when the coefficient α is constant. In this case the following relations hold good:

$$\eta(R(X,Y)Z)=\alpha^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \qquad ...(2.12)$$
$$S(X,\xi)=(n-1)\alpha^2\eta(X). \qquad ...(2.13)$$

Putting $Z=\xi$ in (2.1) and then using (2.12), we get

$$-\alpha^2\{g(Y,W)\eta(X)-g(X,W)\eta(Y)\}= -\frac{b}{a}[S(X,W)\eta(Y)-S(Y,W)\eta(X)] +\frac{r}{n}\left(\frac{1}{n-1}+\frac{b}{a}\right)\{g(X,W)\eta(Y)-g(Y,W)\eta(X)\} \qquad ...(2.14)$$

Again putting $X=\xi$ in (2.14) we get by virtue of (2.13) that

$$S(Y,W)=\left[-\left(\frac{a}{b}\right)\alpha^2+\left(\frac{a+b(n-1)}{n(n-1)b}\right)r\right]g(Y,W)+ \left[-\left(\frac{a+b(n-1)}{b}\right)\alpha^2+\left(\frac{a+b(n-1)}{n(n-1)b}\right)r\right]\eta(Y)\eta(W) \qquad ...(2.15)$$

Hence we can stat the following:

Theorem 2.3. A pseudo $\widetilde{W}_2$ flat LP-Sasakian manifold $M\,(n>1)$ with a constant coefficient α is an η −Einstein manifold.

Differentiating (2.15) covariantly along X and making use of (1.6), we get

$$(\nabla_X S)(Y,W)=dr(X)\left(\frac{a+b(n-1)}{n(n-1)b}\right)+ \alpha\left[-\left(\frac{a+b(n-1)}{b}\right)\alpha^2+ \left(\frac{a+b(n-1)}{n(n-1)b}\right)r\right]\{\Omega(X,Y)\eta(W)+\Omega(X,W)\eta(Y)\},$$

Replacing W by Z in the above equation, we get

$(\nabla_X S)(Y,Z) = dr(X)\left(\frac{a+b(n-1)}{n(n-1)b}\right)[g(Y,Z) + \eta(Y)\eta(Z)]$
$+\alpha\left[-\left(\frac{a+b(n-1)}{b}\right)\alpha^2 + \left(\frac{a+b(n-1)}{n(n-1)b}\right)r\right]\{\Omega(X,Y)\eta(Z) + \Omega(X,Z)\eta(Y)\},$

This implies that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = dr(X)\left(\frac{a+b(n-1)}{n(n-1)b}\right)[g(Y,Z) + \eta(Y)\eta(Z)] - dr(Y)\left(\frac{a+b(n-1)}{n(n-1)b}\right) + \alpha\left[-\left(\frac{a+b(n-1)}{b}\right)\alpha^2 + \left(\frac{a+b(n-1)}{n(n-1)b}\right)r\right]\{\Omega(X,Z)\eta(Y) - \Omega(Y,Z)\eta(X)\}. \quad \ldots(2.16)$$

On the other hand, in our case, since we have $(\nabla_W C)(X,Y)Z = 0$, we get $div\, C = 0$, where "div" denotes the divergence. So for $n > 1$, $div\, C = 0$ gives

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{a}\left\{\frac{a+b(n-1)}{n(n-1)} - \frac{b}{2}\right\}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad \ldots(2.17)$$

It follows from (2.16) and (2.17) that

$$\frac{1}{a}\left\{\frac{a+b(n-1)}{n(n-1)} - \frac{b}{2}\right\}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] = dr(X)\left(\frac{a+b(n-1)}{n(n-1)b}\right)[g(Y,Z) + \eta(Y)\eta(Z)] - dr(Y)\left(\frac{a+b(n-1)}{n(n-1)b}\right)[g(X,Z) + \eta(X)\eta(Z)] + \alpha\left[-\left(\frac{a+2b(n-1)}{b}\right)\alpha^2 + \left(\frac{a+2b(n-1)}{n(n-1)b}\right)r\right]\{\Omega(X,Z)\eta(Y) - \Omega(Y,Z)\eta(X)\}. \quad \ldots(2.18)$$

If r is constant, then from (2.18) we obtain

$$r = n(n-1)\alpha^2. \quad \ldots(2.19)$$

Now substituting (2.15) and (2.19) in (2.1) we get

$$'R(X,Y,Z,W) = \alpha^2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],$$

Which show that the manifold is of constant curvature.

Thus we can state the following:

Theorem 2.4. In a pseudo $\widetilde{W}_2$ flat LP-Sasakian manifold M $(n > 1)$ with a constant coefficient α, if the scalar curvature r is constant, then M is of constant curvature.

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