Sci Vis 18 (1), 11—15 (2018)

SCIENCE VISION

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Research Article

Certain results on Ricci solitons in α-Kenmotsu manifolds

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In this paper, we study some curvature problems of Ricci solitons in α-Kenmotsu manifold. It is shown that a symmetric parallel second order-covariant tensor in a α-Kenmotsu manifold is a constant multiple of the metric tensor. Using this result, it is shown that if ($\mathcal{L}_V g$ + 2S) is parallel where V is a given vector field, then the structure (g, V, λ) yield a Ricci soliton. Further, by virtue of this result, Ricci solitons for n-dimentional α-Kenmotsu manifolds are obtained. In the last section, we discuss Ricci soliton for 3-dimentional α-Kenmotsu manifolds.

2010 Mathematical Subject Classification: 53C25, 53C10, 53C44

Key words: Ricci soliton, α-Kenmotsu manifold, Einstein manifold.

Received 07 November 2017
Accepted 06 January 2018

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https://doi.org/10.33493/scivis.18.01.02

Introduction

A Ricci soliton are the natural generalization of Einstein metric and are defined on a Riemannian manifold. On the manifold M, a Ricci soliton is a triple[1] (g, V, λ) with a Riemannian metric g, a vector field V and a real scalar λ such that

$$(\mathcal{L}_V g)(X,Y) + 2\,S(X,Y) + 2\lambda\, g(X,Y) = 0, \qquad \ldots(1)$$

for any vector fields X, Y on $\chi(M)$ where S is the Ricci tensor and $\mathcal{L}_V$ denotes the Lie derivative operator along the vector field V. The metric satisfying (1) are very interesting in the field of physics and are often referred as quasi-Einstein.[2,3] The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.[4]

Das[5] studied second order parallel tensor on an almost contact metric manifold and found that on an α-K-contact manifold (α being non-zero real constant) a second order symmetric parallel tensor is a constant multiple of the associative positive definite Riemannian metric tensor. It is also proved that in an α-Sasakian manifold there is no non-zero parallel 2-form. The study of Ricci solitons in K-contact manifolds was started by Sharma[6] and in the continuation of this Ghosh, Sharma and Cho[7] studied gradient Ricci soliton of a non-Sasakian (k, μ) -contact manifold. Generally, in a P-Sasakian manifold the structure vector field ξ is not killing, that is $(\mathcal{L}_V g) \neq 0$ but in K-contact manifold ξ is a killing vector field, that is $(\mathcal{L}_V g) = 0$. Recently, De[8] have studied Ricci soliton in P-Sasakian, Barua and De[9] have studied Ricci soliton in Riemannian manifolds. Since then several other studied Ricci soliton have been published in various contact manifolds: Eisenhart problem to Ricci soliton in f-Kenmotsu manifold,[10] Eta-Ricci solitons on para-Kenmotsu manifolds,[11] on contact and Lorentzian manifolds,[10,12,13] on Sasakian manifold,[14,15] α -Sasakian manifold,[16] on Kenmotsu manifold,[17] etc.

Motivated by above studies, in this paper we treat Ricci soliton in α-Kenmotsu manifolds. The paper is structured as follows. After

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introduction, section 2 is a brief review of α-Kenmotsu manifold. Section 3, is devoted to the study of parallel symmetric second order tensor in α-Kenmotsu manifold and Ricci soliton in α-Kenmotsu manifolds. In this section, we obtain a relation between symmetric parallel second order covariant tensor and metric tensor in α-Kenmotsu manifold. In the second problem of this section we studied the necessary and sufficient condition of a Ricci semi-symmetric α-Kenmotsu manifold and η-Einstein manifold. Section 4 is devoted to study Ricci soliton in 3-dimensional α-Kenmotsu manifold.

α-Kenmotsu manifold

An n-dimensional real C^∞-manifold M is said to almost contact structure (φ,ξ,η) if it admits a (1, 1) tensor field φ, a contravariant vector field ξ and a 1-form η which satisfy

$$\eta(\xi)=1,\ \varphi^2X=-X+\eta(X)\xi, \qquad \text{... (2)}$$

which implies

$$\varphi(\xi)=0,\ \ \eta(\varphi X)=0, \qquad \text{...(3)}$$

for all vector field X,Y on $\chi(M)$, where $\chi(M)$ is the Lie algebra of C^∞ vector fields on M. An n-dimensional real C^∞-manifold M equipped with almost contact structure (φ,ξ,η) is called almost contact manifold[18].

An almost contact manifold M with metric tensor g which satisfies the condition

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \qquad \text{...(4)}$$

and $g(X,\xi)=\eta(X)$, ...(5)

is called almost contact metric manifold M (φ,ξ,η,g).

An almost contact metric manifold M is said to be almost α-Kenmotsu manifold if

$$d\eta=0, \qquad \text{and} \qquad d\Phi=2\alpha\,\eta\wedge\Phi,$$

where Φ is a fundamental 2-form defined as $\Phi(X,Y)=g(\varphi X,Y)$ and α being a non-zero real constant.[17] Moreover, if an almost α-Kenmotsu manifold M satisfies the following relations

$$(\nabla_X\varphi)Y=-\alpha\{g(X,\varphi Y)\xi+\eta(Y)\varphi X\}, \qquad \text{...(6)}$$

and $(\nabla_X\xi)=\alpha\{X-\eta(X)\xi\}$, ...(7)

then it is called α-Kenmotsu manifold.[17,18,19]

On an α-Kenmotsu manifold M, the following relations hold[20,21,22]

$$R(X,Y)\xi=\alpha^2\{\eta(X)Y-\eta(Y)X\}, \qquad \text{...(8)}$$

$$R(\xi,X)Y=\alpha^2\{\eta(Y)X-g(X,Y)\xi\}, \qquad \text{...(9)}$$

$$\eta(R(X,Y)Z)=\alpha^2\{g(X,Y)\eta(Z)-g(Y,Z)\eta(X)\}, \qquad \text{...(10)}$$

$$S(X,\xi)=-\alpha^2(n-1)\eta(X), \qquad \text{...(11)}$$

$$S(\xi,\xi)=-\alpha^2(n-1), \qquad \text{...(12)}$$

$$Q\xi=-\alpha^2(n-1)\xi, \qquad \text{...(13)}$$

$$(\nabla_X\eta)Y=\alpha\{g(X,Y)-\eta(X)\eta(Y)\}, \qquad \text{...(14)}$$

for all vector fields X,Y,Z on $\chi(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor of type (0, 2) and Q is the Ricci operator defined as $S(X,Y)=g(QX,Y)$.

Parallel symmetric second order tensors and Ricci solitons in α-Kenmotsu manifolds

Let h denote a (0, 2) type symmetric tensor field which is parallel with respect to ∇ that is $\nabla h=0$. Then it follows that[14, 23]

$$\nabla^2h(X,Y;Z,W)-\nabla^2h(X,Y;W,Z)=0, \qquad \text{...(15)}$$

which gives

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0. \qquad \text{... (16)}$$

Taking $Z=W=\xi$ in (16) and using (8), we have

$$\alpha^2\{\eta(X)h(Y,\xi)-\eta(y)h(X,\xi)\}=0. \qquad \text{...(17)}$$

Since α is non-zero, so by taking $X=\xi$ in (17) and by the symmetry of h, we have

$$h(Y,\xi)=\eta(Y)h(\xi,\xi). \qquad \text{...(18)}$$

Differentiating (18) covariantly with respect to X, we have

$$(\nabla_Xh)(Y,\xi)+h(\nabla_XY,\xi)+h(Y,\nabla_X\xi)$$
$$=(\nabla_X\eta)(Y)h(\xi,\xi)+\eta(\nabla_XY)h(\xi,\xi)$$
$$+\eta(Y)(\nabla_Xh)(\xi,\xi)+2\eta(Y)h(\nabla_X\xi,\xi). \qquad \text{...(19)}$$

By using (7), (14), (18) and the parallel condition $\nabla h=0$ in (19), we have

$$h(X,Y)=g(X,Y)h(\xi,\xi).$$

The above equation implies that $h(\xi,\xi)$ is a constant, via (18). So we have the following theorem.

Theorem 1. *A symmetric parallel second order covariant tensor in an α-Kenmotsu manifold is a constant multiple of the metric tensor.*

Corollary 1. *A locally Ricci symmetric ($\nabla S=0$) α-Kenmatsu manifold is an Einstein manifold.*

Remark 1. *The following statements for α-Kenmatsu manifold are equivalent*

(i) Einstein,

(ii) locally Ricci symmetric,

(iii) Ricci semi-symmetric, that is $R\cdot S=0$.

The implication $(i)\to(ii)\to(iii)$ is trivial. Now we prove that the implication $(iii)\to(i)$ in more general frame work of α-Kenmotsu manifold. Since $R\cdot S=0$, means exactly (16) with h

replaced by S, that is

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \qquad \ldots(21)$$

Taking $R\cdot S = 0$ and putting $X = \xi$ in (21), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \qquad \ldots(22)$$

In view of (9) and $\alpha \neq 0$, the above equation becomes

$$\eta(U)S(Y,V) - g(Y,V)S(\xi,V) + \eta(V)S(U,Y) - g(Y,V)S(U,\xi) = 0. \qquad \ldots(23)$$

Putting $U = \xi$ in (23) and by using (3), (11) and (12), we obtain

$$S(Y,V) = -\alpha^2(n-1)g(Y,V).$$

This lead the following theorem.

Theorem 2. *A Ricci semi-symmetric α-Kenmotsu manifold is an Einstein manifold.*

Corollary 2. *If on an α-Kenmotsu manifold the tensor field $(\mathcal{L}_V g + 2S)$ is parallel, then (g,V,λ) gives a Ricci soliton.*

Proof. A Ricci soliton in α-Kenmotsu manifold is defined by (1). Thus $(\mathcal{L}_V g + 2S)$ is parallel. By theorem (1) it is clear that if an α-Kenmotsu manifold admits a symmetric parallel (0, 2) tensor, then the tensor is a constant multiple of the metric tensor. Hence $(\mathcal{L}_V g + 2S)$ is a constant multiple of metric tensor g that is $(\mathcal{L}_V g + 2S)(X,Y) = g(X,Y)h(\xi,\xi)$, where $h(\xi,\xi)$ is a non zero constant. It is the application of the theorem (1) to Ricci soliton.

Theorem 3. *If a metric g in an α-Kenmotsu manifold is a Ricci soliton with $V = \xi$ then it is η-Einstein.*

Proof. Putting $V = \xi$ in (1), we have

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\lambda\, g(X,Y) = 0, \qquad \ldots(24)$$

where

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi).$$
$$= 2\alpha\{g(X,Y) - \eta(X)\eta(Y)\}. \qquad \ldots(25)$$

Substituting (25) in (24) and by use of (7), we obtain

$$S(X,Y) = -(\alpha+\lambda)g(X,Y) + \alpha\,\eta(X)\eta(Y).$$

Hence the result.

Theorem 4. *A Ricci soliton (g,ξ,λ) in an n-dimentional α-Kenmotsu manifold can not be steady but is shrinking.*

Proof. In the Linear Algebra either the vector field $V \in Span\,\xi$ or $V \perp \xi$. However, the second case seems to be complex to analyse in practice. For this reason, we investigate for the case $V = \xi$.

By a simple computation of $(\mathcal{L}_V g + 2S)$, we obtain

$$(\mathcal{L}_\xi g)(X,Y) = 0. \qquad \ldots(26)$$

$$h(\xi,\xi) = -2\lambda, \qquad \ldots(27)$$

where $h(\xi,\xi) = (\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi). \qquad \ldots(28)$

Using (12) and (26) in above equation, we get

$$h(\xi,\xi) = 2\alpha^2(n-1). \qquad \ldots(29)$$

Equating (27) and (29), we have

$$\lambda = -\alpha^2(n-1).$$

Since α is some non-zero scalar function, we have $\lambda \neq 0$, that is Ricci soliton in an n-dimensional α-Kenmotsu manifold cannot be steady but is shrinking because $\lambda < 0$.

Theorem 5. *If an n-dimensional α-Kenmotsu manifold is η-Einstein then the Ricci solitons in α-Kenmotsu manifold that is (g,ξ,λ) where $\lambda = -\alpha^2(n-1)$ with varying scalar curvature cannot be steady but it is expending.*

Proof. The proof consists of three parts.

(i) We prove α-Kenmotsu manifold is η-Einstein,

(ii) We prove the Ricci soliton in α-Kenmotsu manifold is consisting of varying scalar curvature,

(iii) We find that the Ricci soliton in α-Kenmotsu manifold is expending.

First we prove that the α-Kenmotsu manifold is η-Einstein: the metric g is called η-Einstein if there exists two real function a and b such that the Ricci tensor of g is given by the general equation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \qquad \ldots(30)$$

Let $e_i, (i = 1,2,\ldots n)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (30) and taking summation over i, we get

$$r = a\,n + b. \qquad \ldots(31)$$

Again putting $X = Y = \xi$ in (30) then by use of (12), we have

$$a + b = -\alpha^2(n-1). \qquad \ldots(32)$$

Then from (31) and (32), we have

$$a = \left(\alpha^2 + \frac{r}{n-1}\right),\ b = -\left(n\,\alpha^2 + \frac{r}{n-1}\right). \qquad \ldots(33)$$

Substituting the value of a and b from (33) in (30), we have

$$S(X,Y) = \left(\alpha^2 + \frac{r}{n-1}\right)g(X,Y) - \left(n\,\alpha^2 + \frac{r}{n-1}\right)\eta(X)\eta(Y), \qquad \ldots(34)$$

the above equation shows that α-Kenmotsu manifold is η-Einstein manifold.

Now, we have to show that the scalar curvature r is not a constant and it is varying.

For an n-dimensional α-Kenmotsu manifolds the symmetric parallel covariant tensor $h(X,Y)$ of type (0, 2) is given by

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \quad \dots(35)$$

By using (25) and (34) in (35), we have

$$h(X,Y) = 2\left\{\alpha(\alpha+1) + \frac{r}{n-1}\right\} g(X,Y) - 2\left\{\alpha(n\alpha + 1) + \frac{r}{n-1}\right\}\eta(X)\eta(Y). \quad \dots(36)$$

Differentiating (36) covariantly with respect to Z and using (14), we have

$$(\nabla_Z h)(X,Y) = 2\left\{(Z\alpha)(\alpha+1) + \alpha(Z\alpha) + \frac{\nabla_Z r}{n-1}\right\} g(X,Y) -2\left\{(Z\alpha)(n\alpha+1) + n\,\alpha(Z\alpha) + \frac{\nabla_Z r}{n-1}\right\}\eta(X)\eta(Y) -2\left\{\alpha(n\alpha+1) + \frac{r}{n-1}\right\}\alpha\{g(Z,X) - \eta(Z)\eta(X) +g(Z,Y) - \eta(Z)\eta(Y)\}. \quad \dots(37)$$

By substituting $Z = \xi$ and $X = Y \in (Span)^\perp$ in (37) and by using $\nabla h = 0$, we have

$$\nabla_\xi r = -(n-1)\nabla_\xi\{\alpha(\alpha+1)\}. \quad \dots(38)$$

On integrating (38), we have

$$r = -(n-1)\alpha(\alpha+1) + c, \quad \dots(39)$$

where c is some integral constant. Thus from (39), we have r is a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in α-Kenmotsu manifold:

From (1), we have $h(X,Y) - 2\lambda\, g(X,Y)$ then putting $X = Y = \xi$, we have

$$h(\xi,\xi) = -2\lambda. \quad \dots(40)$$

On putting $X = Y = \xi$ in (36), we have

$$h(\xi,\xi) = -2(n-1)\alpha^2. \quad \dots(41)$$

Equating (40) and (41), we have

$$\lambda = (n-1)\alpha^2.$$

This show that $\lambda > 0$, $\forall\ n > 1$ and hence Ricci soliton in an α-Kenmotsu manifold is expending.

Theorem 6. *If a Ricci soliton (g,ξ,λ) where $\lambda = 2\alpha^2$ of 3-dimensional α-Kenmotsu manifold with varying scalar curvature cannot be steady but it is expending.*

Proof. The proof consists of three parts.

(i) We prove that the Riemannian curvature tensor of 3-dimensional α-Kenmotsu manifold is η-Einstein,

(ii) We prove that the Ricci soliton in 3-dimensional α-Kenmotsu manifold is consisting of varying scalar curvature,

(iii) We prove that find that the Ricci soliton in a 3-dimentional α-Kenmotsu manifold is expending.

The Riemannian curvature tensor of 3-dimensional α-Kenmotsu manifold is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \quad -\frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \quad \dots(42)$$

Putting $Z = \xi$ in (42) and by using (8) and (11), we have

$$\alpha^2\{\eta(X)Y - \eta(Y)X\} = \eta(Y)QX - \eta(X)QY - \left(2\alpha^2 + \frac{r}{2}\right)\{\eta(Y)X - \eta(X)Y\}. \quad \dots(43)$$

Again putting $Y = \xi$ in (43) and by using (2), (3) and (13), we get

$$QX = \left(\alpha^2 + \frac{r}{2}\right)X - \left(3\alpha^2 + \frac{r}{2}\right)\eta(X)\xi. \quad \dots(44)$$

By taking an inner product with Y in (44), we have

$$S(X,Y) = \left(\alpha^2 + \frac{r}{2}\right)g(X,Y) - \left(3\alpha^2 + \frac{r}{2}\right)\eta(X)\eta(Y). \quad \dots(45)$$

It shows that 3-dimensional α-Kenmotsu manifold is η-Einstein manifold.

Now, we have to show that the scalar curvature r is not a constant that is r is varying

We have

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \quad \dots(46)$$

By using (25) and (45) in (46), we have

$$h(X,Y) = 2\left\{\alpha(\alpha+1) + \frac{r}{2}\right\} g(X,Y) - 2\left\{\alpha(3\alpha + 1) + \frac{r}{2}\right\}\eta(X)\eta(Y). \quad \dots(47)$$

Differentiating above equation with respect to Z, we have

$$(\nabla_Z h)(X,Y) = 2\left\{(Z\alpha)(\alpha+1) + \alpha(Z\alpha) + \frac{\nabla_Z r}{2}\right\} g(X,Y) -2\left\{(Z\alpha)(3\alpha+1) + \alpha(3Z\alpha) + \frac{r}{2}\right\}\eta(X)\eta(Y) -2\left\{\alpha(3\alpha+1) + \frac{r}{2}\right\}\{(\nabla_Z\eta)(X)\eta(Y) + \eta(X)(\nabla_Z\eta)(Y)\}. \quad \dots(48)$$

By substituting $Z = \xi$ and $X = Y \in (Span)^\perp$ in (48) and by using $\nabla h = 0$, we have

$$\nabla_\xi r = -2\nabla_\xi\{\alpha(\alpha+1)\}. \quad \dots(49)$$

On integrating (49), we have

$$r = -2\alpha(\alpha+1) + c, \quad \dots(50)$$

where c is some integral constant. Thus from (50), we have r is a varying scalar curvature.

Finally we have to check the nature of the Ricci soliton (g,ξ,λ) in 3-dimensional α-Kenmotsu manifold.

From (1), we have $h(X,Y) - 2\lambda\, g(X,Y)$ then putting $X = Y = \xi$, we have

$$h(\xi,\xi) = -2\lambda. \quad \dots(51)$$

On putting $X = Y = \xi$ in (47), we have

$$h(\xi,\xi) = -4\alpha^2. \quad \dots(52)$$

Equating (51) and (52), we have

Equating (51) and (52), we have
$\lambda = 2\alpha^2$.
This show that $\lambda > 0$ and hence Ricci soliton in an α-Kenmotsu manifold is expending.

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